## THE DESIGN AND ANALYSIS OF COMPUTER ALGORITHMS

Alfred V. Aho<br>Bell Laboratories<br>John E Hopcroft<br>Cornell University<br>Jeffrey D. Ullman<br>Princeton University

Addison-Westey Publishing Company
Reading, Massachusetts - Menlo Park, California
London • Amsterdam • Don Mills, Ontario • Sydney

## CONTENTS

1 Models of Computation
1.1 Algorithms and their complexity ..... 2
1.2 Random access machines ..... 5
1.3 Computational complexity of RAM programs ..... 12
1.4 A stored program model. ..... 15
15 Abstractions of the RAM ..... 19
1.6 A primitive model of computation: the Turing machine ..... 25
1.7 Relationship between the Turing machine and RAM models ..... 31
1.8 Pidgin ALGOL - a high-level language ..... 33
2 Design of Efficient Algorithms
2.1 Data structures: lists, queues, and stacks ..... 44
2.2 Set representations. ..... 49
2.3 Graphs. ..... 50
2.4 Trees ..... 52
2.5 Recursion. ..... 55
2.6 Divide-and-conquer ..... 60
2.7 Balancing ..... 65
2.8 Dynamic programming ..... 67
2.9 Epilogue. ..... 69
3 Sorting and Order Statistics
3.1 The sorting problem. ..... 76
3.2 Radix sorting. ..... 77
3.3 Sorting by comparisons ..... 86
3.4 Heapsort-an $O(n \log \mathrm{n})$ comparison sort. ..... 87
3.5 Quicksort-an $O(n \log n)$ expected time sort ..... 92
3.6 Order statistics ..... 97
3.7 Expected time for order statistics ..... 100
4 Data Structures for Set Manipulation Problems
4.1 Fundamental operations on sets ..... 108
4.2 Hashing. ..... Ill
4.3 Binary search ..... 113
4.4 Binary search trees ..... 115
4.5 Optimal binary search trees. ..... 119
4.6 A simple disjoint-set union algorithm ..... 124
4.7 Tree structures for the UNION-FIND problem ..... 129
4.8 Applications and extensions of the UNION-FIND algorithm. ..... 139
4.9 Balanced tree schemes ..... 145
4.10 Dictionaries and priority queues. ..... 148
4.11 Mergeable heaps. ..... 152
4.12 Concatenable queues ..... 155
4.13 Partitioning ..... 157
4.14 Chapter summary. ..... 162
5 Algorithms on Graphs
5.1 Minimum-cost spanning trees. ..... 172
5.2 Depth-first search. ..... 176
5.3 Biconnectivity ..... 179
5.4 Depth-first search of a directed graph ..... 187
5.5 Strong connectivity .....  189
5.6 Path-finding problems. ..... 195
5.7 A transitive closure algorithm ..... 199
5.8 A shortest-path algorithm ..... 200
5.9 Path problems and matrix multiplication ..... 201
5.10 Single-source problems. ..... 207
5.11 Dominators in a directed acyclic graph: putting the concepts together ..... 209
6 Matrix Multiplication and Related Operations
6.1 Basics ..... 226
6.2 Strassen's matrix-multiplication algorithm. ..... 230
6.3 Inversion of matrices ..... 232
6.4 LUP decomposition of matrices. ..... 233
6.5 Applications of LUP decomposition. ..... 240
6.6 Boolean matrix multiplication ..... 242
7 The Fast Fourier Transform and its Applications
7.1 The discrete Fourier transform and its inverse. ..... 252
7.2 The fast Fourier transform algorithm ..... 257
7.3 The FFT using bit operations ..... 265
7.4 Products of polynomials. ..... 269
7.5 The Schonhage-Strassen integer-multiplication algorithm. ..... 270
8 Integer and Polynomial Arithmetic
8.1 The similarity between integers and polynomials ..... 278
8.2 Integer multiplication and division ..... 279
8.3 Polynomial multiplication and division ..... 286
8.4 Modular arithmetic. ..... 289
8.5 Modular polynomial arithmetic and polynomial evaluation ..... 292
8.6 Chinese remaindering. ..... 294
8.7 Chinese remaindering and interpolation of polynomials ..... 298
8.8 Greatest common divisors and Euclid's algorithm ..... 300
8.9 An asymptotically fast algorithm for polynomial GCD's. ..... 303
8.10 Integer GCD's. ..... 308
8.11 Chinese remaindering revisited ..... 310
8.12 Sparse polynomials ..... 311
9 Pattern-Matching Algorithms
9.1 Finite automata and regular expressions ..... 318
9.2 Recognition of regular expression patterns. ..... 326
9.3 Recognition of substrings ..... 329
9.4 Two-way deterministic pushdown automata ..... 335
9.5 Position trees and substring identifiers ..... 346
10 NP-Complete Problems
10.1 Nondeterministic Turing machines ..... 364
10.2 The classes $0>$ and $J f \&$ ..... 372
10.3 Languages and problems ..... 374
10.4 NP-completeness of the satisfiability problem ..... 377
10.5 Additional NP-complete problems ..... 384
10.6 Polynomial-space-bounded problems. ..... 395
11 Some Provably Intractable Problems
11.1 Complexity hierarchies ..... 406
11.2 The space hierarchy for deterministic Turing machines ..... 407
11.3 A problem requiring exponential time and space ..... 410
11.4 A nonelementary problem ..... 419

## CONTENTS

12 Lower Bounds on Numbers of Arithmetic Operations
12.1 Fields ..... 428
12.2 Straight-line code revisited ..... 429
12.3 A matrix formulation of problems ..... 432
12.4 A row-oriented lower bound on multiplications ..... 432
12.5 A column-oriented lower bound on multiplications. ..... 435
12.6 A row-and-column-oriented bound on multiplications ..... 439
12.7 Preconditioning. ..... 442
Bibliography ..... 452
Index ..... 463

